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Linear algebra – long test – answers

Problem 1 (group 1). (a) Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by

$$[\varphi]_{\rm st}^{\rm st} = \begin{bmatrix} 2 & 2 & 0 \\ 5 & 5 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find a basis $\mathcal{A}$ of $\mathbb{R}^3$ consisting of eigenvectors of φ and find the matrix $[\varphi]_{\mathcal{A}}^{\mathcal{A}}$.

By the Laplace expansion with respect to the third column, we get

$$\begin{aligned} \det\left([\varphi]_{\rm st}^{\rm st} - \lambda I\right) &= \det \begin{bmatrix} 2-\lambda & 2 & 0 \\ 5 & 5-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)\det \begin{bmatrix} 2-\lambda & 2 \\ 5 & 5-\lambda \end{bmatrix} \\ &= (1-\lambda)\left((2-\lambda)(5-\lambda) - 10\right) \\ &= (1-\lambda)(\lambda^2 - 7\lambda) \\ &= -\lambda(\lambda-1)(\lambda-7), \end{aligned}$$

which is zero exactly for $\lambda = 0, 1, 7$. These are the eigenvalues of φ.

For each eigenvalue λ, we find the corresponding eigenspace by solving the system of equations given by the coefficient matrix $[\varphi]_{\rm st}^{\rm st} - \lambda I$. For example, for $\lambda = 7$ we solve

$$\left[\begin{array}{ccc|c} -5 & 2 & 0 & 0 \\ 5 & -2 & 0 & 0 \\ 1 & 0 & -6 & 0 \end{array}\right],$$

find the general solution $x_1 = 6x_3$, $x_2 = 15x_3$ and a basis of solutions consisting of just one vector $(6, 15, 1)$. Similarly, we obtain $(-1, 1, 1)$ for $\lambda = 0$ and $(0, 0, 1)$ for $\lambda = 1$.

We found three vectors, so the required basis indeed exists. We may take the basis $\mathcal{A}$: $(6, 15, 1)$, $(-1, 1, 1)$, $(0, 0, 1)$. In this basis, we have

$$[\varphi]_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues on the diagonal need to be in the same order as eigenvectors in the basis $\mathcal{A}$. For example, $\varphi(6, 15, 1) = (42, 105, 7) = 7 \cdot (6, 15, 1)$, so the coordinates of $\varphi(6, 15, 1)$ in $\mathcal{A}$ are $(7, 0, 0)$ (the first column of $[\varphi]_{\mathcal{A}}^{\mathcal{A}}$).

Problem 1 (group 2). (a) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & 0 \\ 4 & 4 & 5 \end{bmatrix}.$$

Find all eigenvalues of A and the corresponding eigenspaces. Decide whether A is diagonalizable. If it is, find a diagonal matrix D and an invertible matrix C such that $A = CDC^{-1}$.

By the Laplace expansion with respect to the second row, we get

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & -3-\lambda & 0 \\ 4 & 4 & 5-\lambda \end{bmatrix} \\ &= (-3-\lambda)\det \begin{bmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{bmatrix} \\ &= (-3-\lambda)\left((1-\lambda)(5-\lambda) - 12\right) \\ &= (-3-\lambda)(\lambda^2 - 6\lambda - 7) \\ &= -(\lambda+3)(\lambda+1)(\lambda-7), \end{aligned}$$

which is zero exactly for $\lambda = -3, -1, 7$. These are the eigenvalues of A.

For each eigenvalue λ, we find the corresponding eigenspace by solving the system of equations given by the coefficient matrix $A - \lambda I$. For example, for $\lambda = 7$ we solve

$$\left[\begin{array}{ccc|c} -6 & 2 & 3 & 0 \\ 0 & -10 & 0 & 0 \\ 4 & 4 & -2 & 0 \end{array}\right],$$

find the general solution $x_2 = 0$, $x_3 = 2x_1$ and a basis of solutions consisting of just one vector $(1, 0, 2)$. Similarly, we obtain $(1, -5, 2)$ for $\lambda = -3$ and $(-3, 0, 2)$ for $\lambda = -1$.

We know now that the matrix A is diagonalizable, since these three eigenvectors form a basis of $\mathbb{R}^3$. One of the possible choices is:

$$C = \begin{bmatrix} 1 & 1 & -3 \\ 0 & -5 & 0 \\ 2 & 2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The eigenvalues on the diagonal of D need to be in the same order as eigenvectors the columns of C.

Problem 1 (both groups). (b) Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Find all eigenvalues of A and the corresponding eigenspaces. Decide whether A is diagonalizable. If it is, find a diagonal matrix D and an invertible matrix C such that $A = CDC^{-1}$.

Alternatively, given the linear map $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $[\varphi]_{\text{st}}^{\text{st}} = A$, find all eigenvalues of φ and the corresponding eigenspaces. Decide whether there is a basis $\mathcal{A}$ of $\mathbb{R}^3$ such that $[\varphi]_{\mathcal{A}}^{\mathcal{A}}$ is diagonal.

By the Laplace expansion with respect to the second row, we get

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{bmatrix} -1-\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ -1 & 0 & 1-\lambda \end{bmatrix} \\
&= -\lambda \det \begin{bmatrix} -1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} \\
&= -\lambda\left((\lambda+1)(\lambda-1)+1\right) \\
&= -\lambda^3,
\end{aligned}$$

which is zero exactly for $\lambda = 0$. This is the only eigenvalue of A.

The corresponding eigenspace is found by solving the system of equations given by the coefficient matrix A. Actually, it is already solved! We easily find the general solution $x_1 = x_3$ and a basis of solutions $(0, 1, 0)$, $(1, 0, 1)$. Since we did not obtain a basis of $\mathbb{R}^3$, A and φ are not diagonalizable (and such C and $\mathcal{A}$ do not exist).

Note that although x_3 does not appear in the general solution, it is still a free variable and hence we get two basis vector, not just one.

If we only care about diagonalization, things get easier. Once we know there is only one eigenvalue $\lambda = 0$, the answer is obvious – A and φ are not diagonalizable. Otherwise they would have to be identically zero (and they are not).

Problem 2 (both groups). Let

$$u_1 = (1, 0, 1, 1),\ u_2 = (1, 2, 1, -2),\ u_3 = (-1, 0, 1, 0),\ u_4 = (1, -3, 1, -2).$$

- Is the system u_1, u_2, u_3, u_4 orthogonal? Is it orthonormal?
- Justify that u_1, u_2, u_3, u_4 is a basis of $\mathbb{R}^4$.
- Find the second coordinate of $\alpha = (9, -7, 4, 3)$ in this basis.

(a) To investigate orthogonality of the system u_1, u_2, u_3, u_4, we compute

$$u_1 \cdot u_2 = 0,\ u_1 \cdot u_3 = 0,\ u_1 \cdot u_4 = 0,\ u_2 \cdot u_3 = 0,\ u_2 \cdot u_4 = 0,\ u_3 \cdot u_4 = 0.$$

Thus, it is orthogonal. Note that the scalar product is commutative and hence we do not have to check e.g., $u_4 \cdot u_2 = 0$, as this is the same as $u_2 \cdot u_4 = 0$.

To investigate orthonormality of the system u_1, u_2, u_3, u_4, we compute $u_i \cdot u_i$ for each $i = 1, 2, 3, 4$ (orthogonality is already checked). Actually, we compute

$$u_1 \cdot u_1 = 1^2 + 0^2 + 1^2 + 1^2 = 3 \neq 1,$$

and hence the system is not orthonormal. Note that we do not need to check e.g., $u_2 \cdot u_2 = 1$; the system is not orthonormal anyway.

(b) Variant I. Since u_1, u_2, u_3, u_4 is an orthogonal system and all these vectors are non-zero (this second part is also important), it is a basis.

This simple fact was given both in the lecture and the exercise classes, but let us present this argument one more time. Consider a linear combination $a_1u_1 + \ldots + a_4u_4$ which gives us the zero vector $(0, 0, 0, 0)$. Taking the scalar product with itself, we obtain the equalities

$$\begin{aligned} 0 &= (0, 0, 0, 0) \cdot (0, 0, 0, 0) \\ &= (a_1u_1 + \ldots + a_4u_4) \cdot (a_1u_1 + \ldots + a_4u_4) \\ &= a_1^2 u_1 \cdot u_1 + \ldots + a_4^2 u_4 \cdot u_4 \\ &= 3a_1^2 + 10a_2^2 + 2a_3^2 + 15a_4^2, \end{aligned}$$

from which we conclude that $a_1 = a_2 = a_3 = a_4 = 0$. This shows linear independence of our four vectors in $\mathbb{R}^4$, and hence they are a basis.

(b) Variant II. To justify that u_1, u_2, u_3, u_4 is a basis, we may for example put these vectors as rows of a matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & -2 \\ -1 & 0 & 1 & 0 \\ 1 & -3 & 1 & -2 \end{bmatrix}$$

and perform elementary operations on rows until we get the 4×4 identity matrix.

This is quite time-consuming and prone to errors, but correct.

(c) Variant I. Since it is an orthogonal basis, the second coordinate is given by

$$\frac{\alpha \cdot u_2}{u_2 \cdot u_2} = \frac{9 - 14 + 4 - 6}{1^2 + 2^2 + 1^2 + (-2)^2} = -\frac{7}{10}.$$

(c) Variant II. The coordinates can be found in the usual way by solving the system

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 9 \\ 0 & 2 & 0 & -3 & -7 \\ 1 & 1 & 1 & 1 & 4 \\ 1 & -2 & 0 & -2 & 3 \end{array}\right].$$

The result is $(\frac{16}{3}, -\frac{7}{10}, -\frac{5}{2}, \frac{28}{15})$. We are only interested in the second coordinate, so the answer is $-\frac{7}{10}$.

The calculations are a nightmare, but this is still a correct solution.